INTERNATIONAL MATHEMATICS
TOURNAMENT OF TOWNS

SENIOR PAPER: YEARS 11,12

Tournament 39, Northern Spring 2018 (A Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. Aladdin has several gold coins and from time to time he asks the Genie to give him more. On each such occasion the Genie first responds by adding a thousand gold coins and then he takes back a half of the total weight of all Aladdin's gold coins. If after asking the Genie for more gold ten times, is it possible for Aladdin that the number of his gold coins has increased taking into account that each time the Genie takes a half of all Aladdin's gold back and no coin is broken into smaller pieces? (4 points)

2. Do there exist 2018 positive reduced fractions, each with a different denominator, such that the denominator of the difference of any two (after reducing to lowest terms) is less than the denominator of any of the initial 2018 fractions? (5 points)

3. One hundred different numbers are written in the squares of a 10×10 table, one number in each square. For each move one can select a rectangle consisting of some squares, and for each square of that rectangle swap its number with the number in the square opposite to it with respect to the centre of the rectangle (i.e. make a rotation of the rectangle by $180°$). Is it always possible to arrange the numbers in the table taking no more than 99 moves so that the numbers increase from left to right in each row, and from bottom to top in each column? (6 points)

4. An equilateral triangle lying in the plane α is orthogonally projected onto a plane β, which is not parallel to α. The resulting triangle is again orthogonally projected onto a plane γ, and its image is an equilateral triangle again. Prove that

   (a) the angle between the planes α and β is equal to the angle between the planes β and γ. (4 points)

   (b) the plane β intersects the planes α and γ along the lines which are perpendicular to each other. (4 points)

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5. You are travelling to some country and you don't know its language. You know that symbols "!" and "?" stand for addition and subtraction, but you don't know which symbol is for which operation. Each of these two symbols can be written between two arguments, but for subtraction you don't know if the left argument is subtracted from the right or vice versa. For example, $a?b$ could mean any of $a-b$, $b-a$ and $a+b$. You don't know how to write any numbers, but variables and brackets can be used as usual. Given two arguments a and b how can you write for sure an expression that is equal to $20a-18b$? (10 points)

6. Let quadrilateral $ABCD$ be inscribed into a circle S. Let P be the intersection point of the rays BA and CD. Let U and V be the intersection points of the line going through P and parallel to the tangent to S at point D, with the tangents to S at points A and B respectively. Prove that the circumcircle of triangle CUV is tangent to the circle S. (10 points)

7. The King decides to reward a group of n wizards. The wizards are placed in line one after another (so that they can see in the same direction only), each of them wearing either a black or white hat. Each wizard can see the hats of all the wizards in front of him. Starting from the back of the line, each wizard in turn announces a colour (black or white) and a natural number of his choice. The King then counts the number of wizards who nominated the colour of his own hat, and then grants a pay bonus for the same number of days to all the wizards. The wizards are allowed to decide on a common strategy prior to forming the line, but they know that k of them are insane. However, they do not know who of them is insane. An insane wizard tells a white or black colour and a natural number regardless of the common strategy. What is the maximum number of days with bonus pay that can be guaranteed for sure with the common strategy no matter where the insane wizards are placed in the line? (12 points)

A Level Senior Paper Solutions

Edited by Oleksiy Yevdokimov and Greg Gamble

1. **Solution 1.** No, it is not possible. Suppose Aladdin initially has $1000 + x$ gold coins. Then, after asking the Genie for more gold coins, once, Aladdin will have $1000 + x/2$ gold coins, and after asking the Genie for more coins, ten times, he will have $1000 + x/2^{10}$ gold coins. Since no coin is broken into smaller pieces, x must be divisible by 1024. Since Aladdin initially had a positive number of coins, $x > -1000$. Thus, for divisibility by 1024, x must in fact be non-negative, so that $1000+x/2^{10} \leq 1000+x$, and hence, it is not possible that the number of Aladdin's gold coins could have increased.

   Solution 2. No, it is not possible. Going backwards, if Aladdin has x gold coins after asking the Genie once, then he had $2x - 1000$ gold coins before asking the Genie. Similarly, if Aladdin has x gold coins after asking the Genie ten times, then he had

$$\begin{aligned} 2(2(\cdots(2x-1000)\cdots)-1000)-1000 &= 2^{10}x - 1000(2^9 + \cdots + 2 + 1) \\ &= 2^{10}x - 1000(2^{10}-1) \end{aligned}$$

   gold coins, initially.

   Suppose that the number of Aladdin's gold coins has increased. Then

$$\begin{aligned} 2^{10}x - 1000(2^{10}-1) &< x \\ (2^{10}-1)x &< 1000(2^{10}-1) \\ x &< 1000 \end{aligned}$$

   However, $2^{10}x - 1000(2^{10}-1) > 0$, and so we get

$$x > \frac{1000(2^{10}-1)}{2^{10}} = 1000\left(1 - \frac{1}{1024}\right) > 999$$

   which leads to a contradiction since we have $999 < x < 1000$ for an integer x.

2. **Solution 1.** Such 2018 positive reduced fractions do exist. Consider the fractions,

$$\frac{1+q}{q}, \frac{2+q}{2q}, \ldots, \frac{2018+q}{2018q},$$

   where $q = 2018! + 1$. These fractions cannot be reduced since $(q, i) = 1$ for $1 \leq i \leq 2018$. The difference, of any two of the fractions above, can be written as

$$\frac{i+q}{iq} - \frac{j+q}{jq} = \frac{j(i+q) - i(j+q)}{ijq} = \frac{j-i}{ij},$$

   where $1 \leq i, j \leq 2018$. Thus, the denominator of the difference of any two of the fractions is less than q, and hence less than q after reducing to lowest terms. So we are done.

Solution 2. Such 2018 positive reduced fractions do exist. Choose any 2018 positive reduced to lowest terms fractions with numerators $a_1, a_2, \ldots, a_{2018}$ and respective denominators $b_1 > b_2 > \cdots > b_{2018} > 0$. Choose a positive fraction of the form $1/d$ where $d > b_1 b_2$ and $(d, b_1 b_2 \cdots b_{2018}) = 1$. Then add $1/d$ to each of the 2018 chosen fractions, to obtain

$$\frac{a_i}{b_i} + \frac{1}{d_i} = \frac{a_i d + b_i}{b_i d},$$

for each i such that $1 \le i \le 2018$. The 2018 fractions thus obtained satisfy all requirements, since their reduced form denominators are db_i, as $(a_i d + b_i, db_i) = 1$, and the difference of any two of them,

$$\frac{a_i d + b_i}{b_i d} - \frac{a_j d + b_j}{b_j d} = \frac{(a_i d + b_i) b_j - (a_j d + b_j) b_i}{b_i b_j d} = \frac{a_i b_j - a_j b_i}{b_i b_j},$$

has denominator at most $b_1 b_2 < d < db_i$.

Solution 3 (by William Steinberg). Such 2018 positive reduced fractions do exist. Take 2019 primes $p_1 < p_2 < \cdots < p_{2018} < p_{2019}$. They are coprime; so, for each $i < 2019$, there exists b_i such that $b_i p_i \equiv 1 \pmod{p_{2019}}$.

By the Chinese Remainder Theorem, for each $i < 2019$, there exists an a_i satisfying the system of 2019 congruences,

$$\begin{aligned} a_i &\equiv b_i \pmod{p_{2019}} \\ a_i &\equiv 1 \pmod{p_j}, \text{ for } 1 \le j \le 2018. \end{aligned}$$

We will show that the 2018 fractions,

$$\frac{a_i p_i}{p_1 p_2 \cdots p_{2018} p_{2019}},$$

where $1 \le i \le 2018$, satisfy the requirements. Since a_i by design is coprime to each of the primes $p_1, p_2, \ldots, p_{2018}$, the reduced denominator of the ith fraction is

$$\frac{p_1 p_2 \cdots p_{2018} p_{2019}}{p_i}.$$

Now, for $1 \le i, j \le 2018$,

$$a_i p_i \equiv 1 \equiv a_j p_j \pmod{p_{2019}}.$$

So p_{2019} divides $a_i p_i - a_j p_j$. Hence the difference of the ith and jth fractions, $i \ne j$, when reduced to lowest terms, is at most

$$p_1 p_2 \cdots p_{2018} = \frac{p_1 p_2 \cdots p_{2018} p_{2019}}{p_{2019}} < \frac{p_1 p_2 \cdots p_{2018} p_{2019}}{p_i},$$

for $1 \le i \le 2018$.

3. **Solution.** It is always possible. Colour all numbers red. We claim that for each move we can select some rectangle consisting of squares with red numbers only, make a rotation of that rectangle by $180°$ and re-colour one number green keeping the following properties:

   (i) every time a green number is created, it is less than any of the remaining red numbers;

   (ii) the created green numbers increase from left to right in each row, and from bottom to top in each column, with the red numbers following in arbitrary order.

   Note that, at the beginning, the properties hold trivially, since there are no green numbers. Assume that the properties hold before move n. We show that the properties hold after move n. Indeed, let the least of the red numbers remaining be x in the square A. Going down from A along squares with red numbers only, as far as possible, we come to the square B (we are stopped either by a green square or the edge of the table). Then, going left from B along squares with red numbers only as far as possible we come to the square C. In this way, we obtain a rectangle $ABCD$ (it may consist of one square only) with red numbers only in all squares. Make a rotation of $ABCD$ by $180°$. Then x will be in the square C. All numbers left of or below C have already been coloured green and are less than x. Now re-colour x green. This completes move n, and properties (i) and (ii) again hold.

   Since exactly one number is coloured green on each move, and once a number is coloured green, it is not moved again, after 99 moves there will remain one red number only. Since property (ii) is satisfied, the remaining red number can only be at the top right corner of the table, and by property (i) it is already correctly placed. Hence, we are done.

4. **Solution.** Let the planes α and γ intersect the plane β along the lines a and b, forming with β, angles φ and ψ, respectively. For the plane α, rotate through angle φ along the line a, so that α and β coincide. Similarly, for the plane γ, rotate through angle ψ along the line b, so that γ and β coincide. Thus, the projection U' onto the plane β, of a point U in the plane α at distance d from the line a, is at distance $d\cos\varphi$ from the line a. Similarly, the projection V' onto the plane β, of a point V in the plane γ at distance d' from the line b, is at distance $d'\cos\psi$ from the line b. Note that the lines a and b are not parallel. Since an equilateral triangle has been transformed into an equilateral triangle under the two transformations, the composition of these two transformations is an homothety with a fixed point at the intersection of the lines a and b. Call this fixed point O. Let points X and Y lie on the lines a and b, respectively.

   (a) Assume (b) already proved and $\angle XOY = 90°$. Since the composition of these two transformations is an homothety with coefficient $k > 0$, angles are preserved and $\angle XOY$ is transformed into itself. So we get that a point with coordinates (x, y) is transformed into a point with coordinates $(x, y\cos\varphi)$ first and then into a point with coordinates $(x\cos\psi,\ y\cos\varphi) = (xk,\ yk)$ which means $\varphi = \psi$ as required.

(b) Assume the lines a and b are not perpendicular and $\angle XOY < 90^\circ$. Then, after the first transformation, X remains fixed (unmoved) and Y is transformed inside of $\angle XOY$. Furthermore, after the second transformation both points are transformed inside $\angle XOY$. Hence, the angles are not preserved which is a contradiction. Thus, the lines a and b are perpendicular to each other.

5. **Solution.** To facilitate the writing of a linear combination of a and b we first find representations for 0, the sum of two symbols a and b, and the opposite symbol $-a$.

An expression $(a?a)!(a?a)$ is always equal to 0. So we can write 0 now bearing in mind that we mean $(a?a)!(a?a)$.

An expression $(a?0)?(0?b)$ is equal to $a+b$. Similarly to above, we can write $a+b$, bearing in mind that we mean $(a?0)?(0?b)$.

Furthermore, $0?((0!(a!0))?0)$ is always equal to $-a$. Thus, we can represent an expression that is equal to $20a - 18b$ using the operations we have defined above:

$$\underbrace{((\cdots(a+a)+\cdots+a)+a)}_{\text{adding 20 symbols } a} + \underbrace{(-((\cdots(b+b)+\cdots+b)+b))}_{\text{adding 18 symbols } b}.$$

Note. The representations used for 0, $a+b$ and $-a$ are not unique. Other representations can be obtained by replacing "?" with "!" and vice versa.

6. **Solution.** Let UC and VC intersect the tangent to S at point D, at K and L, respectively. Furthermore, let UC and VC intersect S (for the second time) at points X and Y, respectively. Let T be the intersection point of the two tangents to S constructed at points A and B, respectively. Applying Menelaus's theorem to triangle UVT and line BP we get

$$\frac{UP}{PV} \cdot \frac{VB}{BT} \cdot \frac{TA}{AU} = 1.$$

Being tangents to S from a common point, we have $BT = TA$. Since transversals CU, CP and CV from common point C cut parallels KL and UV in the same ratio, $UP/PV = KD/DL$. Hence,

$$\frac{KD \cdot VB}{DL \cdot AU} = 1 \quad \text{or equivalently} \quad \frac{UA}{KD} = \frac{VB}{LD}.$$

By the Tangent-Secant version of the Power of a Point theorem, we have

$$\begin{aligned} UX \cdot UC &= UA^2 \\ KX \cdot KC &= KD^2 \\ VY \cdot VC &= VB^2 \\ LY \cdot LC &= LD^2, \end{aligned}$$

and hence

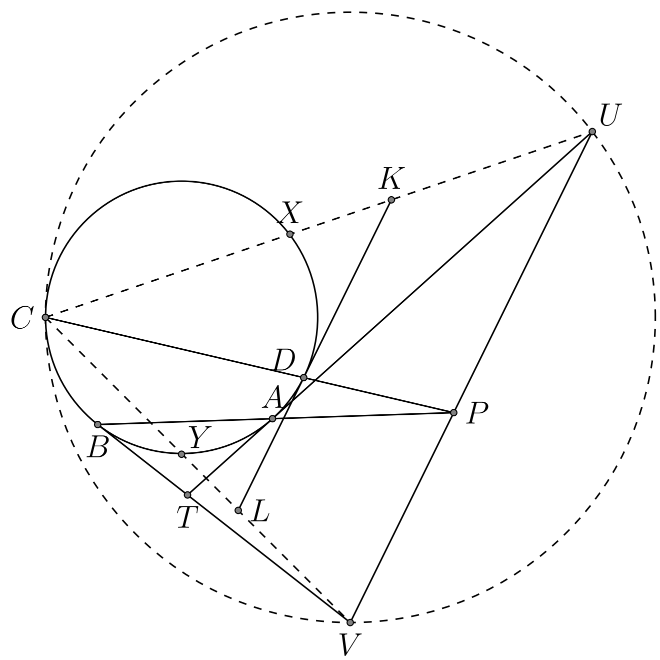
$$\frac{UX \cdot UC}{KX \cdot KC} = \frac{UA^2}{KD^2} = \frac{VB^2}{LD^2} = \frac{VY \cdot VC}{LY \cdot LC}.$$

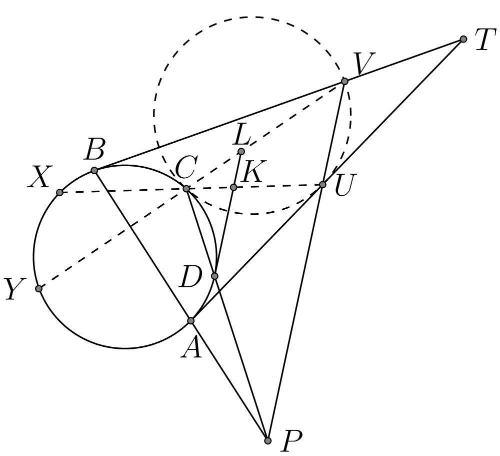
Since transversals from the common point C are cut in the same ratio by parallels KL and UV, $UC/KC = VC/LC$ and hence the above reduces to

$$\frac{UX}{KX} = \frac{VY}{LY}.$$

Therefore, by the converse of Thales Intercept theorem lines XY and UV are parallel. Hence, there exists an homothety with centre at C transforming triangle CXY into triangle CUV. Therefore, their circumcircles touch at point C and we are done.

The two diagrams below show two different cases, where the circle S and the circumcircle of CUV touch internally or touch externally.





7. **Solution.** The maximum number of days is $n - k - 1$. Since k insane wizards can tell their answers regardless of the common strategy, their answers cannot be guaranteed. The correct answer of the first wizard who is not insane also cannot be guaranteed since he does not have any information about the colour of his hat. Therefore, there may be $n-k-1$ correct answers at most. We show that a strategy to achieve $n - k - 1$ correct answers exists.

   Note there are 2^i ways that i wizards can wear either a black or a white hat. So, wizards can arrange between them a coding for the colours of hats of all the wizards in front of each wizard (that also accounts for several wizards being insane), where a wizard, who can see i hats in front of him, announces a number from 1 to 2^i matching a particular combination of black and white hats he can see. For example, for a wizard who can see three wizards wearing hats in front of him, the full set of possible combinations in alphabetical order and their matching numbers are shown below.

BBB	1
BBW	2
BWB	3
BWW	4
WBB	5
WBW	6
WWB	7
WWW	$8 = 2^3$

Depending on the combination of hats each wizard sees in front of him, he announces, if he is not insane, of course, a number matching that combination, e.g. if he sees white and then two black hats in front of him, i.e. WBB, he announces 5. If a wizard announces a number not in the range from 1 to 2^i for some i, other wizards replace such a number with 1. Thus, each wizard with the exception of the last at the back of the line receives information about the colour of his hat from all wizards behind him. However, each wizard has to figure out whose advice he should heed, and whose advice he should ignore.

We begin by calling the wizard at the back, who starts the announcements, the current *Speaker*. The strategy of each wizard, who is not insane, is to announce the colour of his hat he received from the current Speaker. If the next wizard heeds the advice of the wizard behind him, that wizard becomes the current Speaker. Otherwise, the current Speaker is unchanged. Since all wizards can hear all the information announced, each wizard can identify the current Speaker at any time. Note that each wizard who is not insane will be the current Speaker at some moment. Thus, on the way from one non-insane wizard to another non-insane wizard, including the latter one, somebody will take the advice and announce the correct colour of his hat. Hence, there will be at least $n - k - 1$ correct answers, which completes the proof.